On the Solution of MHD Jeffery–Hamel Flow by Weighted Residual Method

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***Abstract*—*This article presents the numerical solution of MHD Hamel-Jeffery flow using weighted residual method under which two cases of trial function were examined. The effect of Hartmann and Reynolds number were discussed. Comparisons were made with the exact solution to check the efficiency and accuracy of the method. The results reveal that the method is very effective and simple in solving nonlinear problems. The results obtained are in polynomial form that makes it possible to evaluate the solution at any point in the region.*

Keywords*— Magneto hydrodynamic, Jeffery-Hamel flow, Convergent channel, Divergent channel, weighted residual method.

I. INTRODUCTION

The incompressible viscous fluid flow of Jeffery Hamel equation through divergent –convergent channels is one of the most applicable cases in fluid mechanics. The mathematical investigation of this problem was proposed by [1] and [2] that is (Jeffery -Hamel flows). It is a special case of two dimensional Navier Stoke equations that flows through a channel with inclined plane walls meeting at a vertex with a source or sink at the vertex and have been extensively study by several researchers such as [3]-[7] while [7] gave exact solution with different values Hartmann and Reynolds number. In this study, we have applied WRM to find the approximate solutions of nonlinear differential equation governing Jeffery Hamel flow, and have made a comparison with the exact solution.

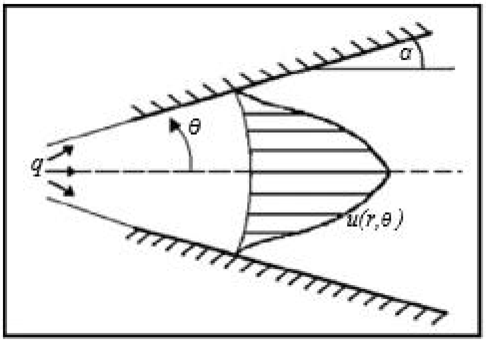

Fig.1: Geometry of the problem.

II. MATHEMATICAL FORMULATION

Consider the steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at the intersection between two rigid plane walls where the angle between them is 2α as shown in fig.1. The rigid walls are known to be convergent if $\alpha < 0$ and divergent if $\alpha > 0$. We assume that the velocity is only along radial direction and depends on r and θ, $v = (u(r,\theta), 0)$ [7]. Using continuity and Navier-Stokes equations in polar coordinates

$$\frac{\rho \partial (r u(r,\theta))}{r \partial r} = 0 \tag{1}$$

$$u(r,\theta)\frac{\partial u(r,\theta)}{\partial r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+ \nu\left[\begin{array}{l}\frac{\partial^2 u(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial u(r,\theta)}{\partial r}\\+\frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2}-\frac{u(r,\theta)}{r^2}\end{array}\right]-\frac{\sigma B_0^2}{\rho r^2}u(r,\theta) \quad (2)$$

$$-\frac{1}{\rho r}\frac{\partial P}{\partial \theta}+\frac{2\nu}{r^2}\frac{\partial u(r,\theta)}{\partial \theta}=0 \quad (3)$$

where B_0 is the electromagnetic induction, σ the conductivity of the fluid, $u(r)$ the velocity along radial direction, P the fluid pressure, ν the coefficient of kinematic viscosity, and ρ the fluid density.

From the continuity equation

$$ru(r,\theta)=f(\theta) \quad (4)$$

Using dimensionless parameters

$$F(x)=\frac{f(\theta)}{U_{\max}}, \qquad x=\frac{\theta}{\alpha} \quad (5)$$

And substituting them into equation 2 and equation 3 and eliminating the pressure term gives third order boundary value problem

$$F'''(x)+2\alpha \operatorname{Re} F(x)F'(x)+(4-Ha)\alpha^2 F'(x)=0 \quad (6)$$

$$F(0)=1, F'(0)=0, F(1)=0$$

Where Reynolds number $\operatorname{Re}=\frac{\alpha U_{\max}}{\nu}$ and Hartmann number $Ha=\sqrt{\frac{\sigma B_0^2}{\rho \nu}}$

III. WEIGHTED RESIDUAL METHOD

We seek for a polynomial of the form

$$w(x,a)=\phi_0(x)+\sum_{i=1}^{N}a_i\phi_i(x) \quad (7)$$

where $\phi_0(x)$ satisfies the given boundary conditions and each $\phi_i(x)$ satisfies the homogenous form of the boundary conditions. The function $w(x,a)$ is then used as an approximation to the exact solution in the equation

$$L(U(x))=Q(x) \quad (8)$$

to give

$$R(x)=L(U(x))-Q(x) \quad (9)$$

The function $R(x)$ is the residual. The idea is to make $R(x)$ as small as possible. One of the methods of minimizing $R(x)$ can now be used. In this work collocation method is employed, where R(x) is set to zero at some points in the interval. The systems of these equations are then solved to determine the parameters a_i. $w(x,a)$ is then considered as the approximate solution. Any polynomial can be used provided it satisfies the above mentioned conditions [8].

IV. APPLICATION OF WRM TO JEFFERY HAMEL FLOW

Case 1

To satisfy the boundary condition of equation (6), we choose the trial function

$$F(x)=(1-x^2)+\sum_{i=1}^{n}a_i(x^{i+1}-x^{i+2}) \quad (10)$$

Substituting equation (10) into equation (6), gives the residual

For specified value of N, we have system of equations where the values of $a's$ were determined for different value of $\alpha, \operatorname{Re}, Ha$ as shown in table 1, 2, 3 and 4.

Case 2

We assume a trial function in term of general polynomial as

$$F(x)=\sum_{i=0}^{n}a_i x^i \quad (11)$$

Now, forcing the trial function to satisfy the boundary condition of (6) and substituting (11) into (6), gives the residual.

For specified value of N, we have system of equations where the values of $a's$ were determined for different values of α, Re, Ha as presented in tables 1, 2, 3 and 4.

Table.1:Exact value of $F''(0)$ *for different values of Hartmann numbers* $\alpha = -5^{\circ}, \mathrm{Re} = 10$

Ha	Exact	Case1	Case 2
0	−1.784546840578866	-1.784545833	-1.784562149
200	−1.576129888848735	-1.576131666	-1.576129929
400	−1.396317528806288	-1.396323798	-1.396317585
600	−1.240534453880220	-1.240544288	-1.240530116
800	−1.105048380482003	-1.105057069	-1.105048232
1000	−0.986793975627043	-0.9867919548	-0.986794141
2000	−0.578631726922245	-0.5782892766	-0.5786319016
3000	−0.354485132165693	-0.3530670988	-0.3544966468
4000	−0.224749102442234	-0.2215235008	-0.2245805322
5000	−0.146513718051115	-0.1411933588	-0.1465914617

Table.2: Exact value of $F''(0)$ *for different values of Hartmann numbers* $\alpha = 5^{\circ}, \mathrm{Re} = 10$

Ha	Exact	Case1	Case2
0	−2.251948602981818	-2.251949092	-2.251948586
200	−1.984606164603458	-1.984607310	-1.984606182
400	−1.754093033347798	-1.754097081	-1.754093096
600	−1.554605992057426	-1.554616910	-1.554605007
800	−1.381369953213575	-1.381394151	-1.381373986
1000	−1.230437181792459	-1.230483653	-1.230437299
2000	−0.712584949074417	-0.7130035006	-0.712480988
3000	−0.431607723269544	-0.4331967690	-0.4317654604
4000	−0.270901503198049	-0.2749486750	-0.2705858652
5000	−0.175043638247544	-0.1833129636	-0.1750445642

Table.3: Exact value of $F''(0)$ *for different values of Reynolds numbers* $\alpha = -5^{\circ}, Ha = 0$

Re	Exact	Case 1	Case 2
10	−1.784546840578866	-1.784545833	-1.784546499
20	−1.588153536993253	-1.588148456	-1.588153486
30	−1.413692023436293	-1.413691117	-1.413692089
40	−1.258993935085547	-1.259043264	-1.258993911
50	−1.121989043572515	-1.122204305	-1.121990317
60	−1.000742927541422	-1.001336328	-1.000873589
70	−0.893474274815973	-0.8947758600	-0.8934742390
80	−0.798567273772039	-0.7981030699	-0.7985670334
90	−0.714567787224302	-0.7187673212	-0.7146484594
100	−0.640177852876257	-0.6467933166	-0.6401814440

Table.4: Exact value of $F''(0)$ *for different values of Reynolds numbers* $\alpha = 5^\circ, Ha = 0$

Re	Exact	Case 1	Case 2
10	−2.251948602981818	-2.251949092	-2.251948586
20	−2.527192232687426	-2.527190036	-2.527193984
30	−2.832629302137010	-2.832591766	-2.832629312
40	−3.169712187544089	-3.169537108	-3.169659978
50	−3.539415645558434	-3.538896072	-3.539415488
60	−3.942140271189139	-3.940963580	-3.942154774
70	−4.377652476579882	-4.375466212	-4.377650892
80	−4.845071824815794	-4.841643728	-4.844197828
90	−5.342911258048444	-5.338392212	-5.342912102
100	−5.869165116815401	-5.864438428	-5.869173060

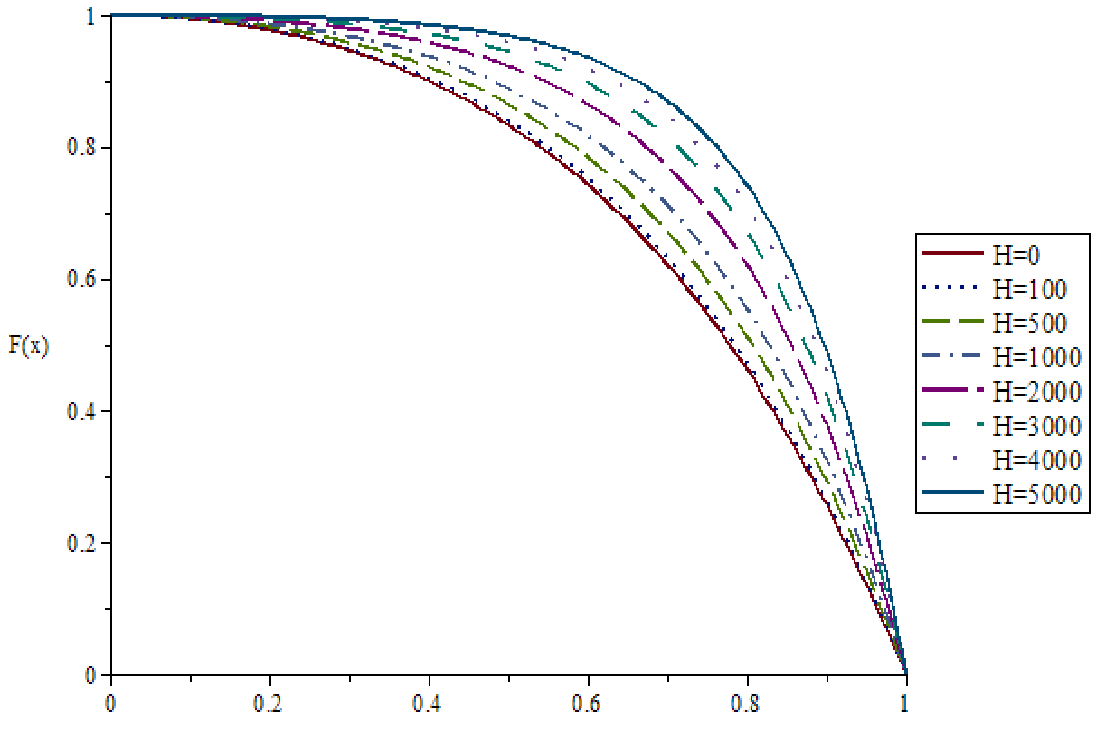

Fig. 2: The WRM solution for velocity F(x) against x for convergent channel, Re=50, α=-5

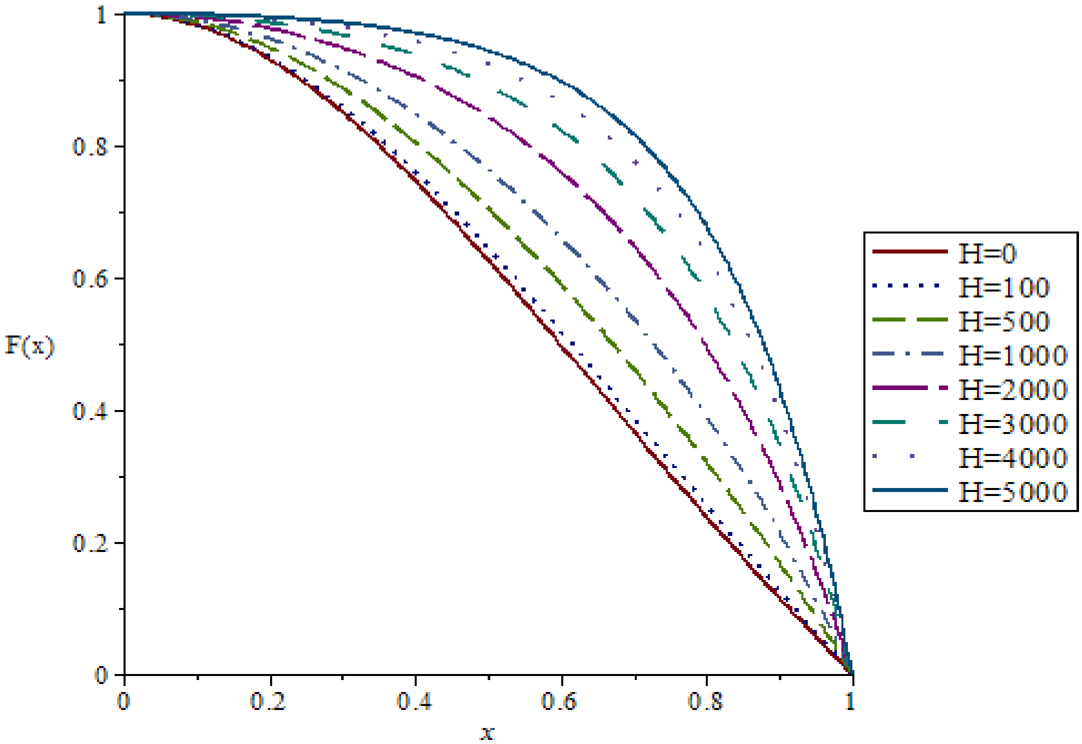

Fig. 3: The WRM solution for velocity F(x) against x for divergent channel, Re=50, α=5

V. DISCUSSION OF RESULTS

Different solutions were gotten by varying parameters α, Re, Ha that give rise to tables and figures as shown. Table 1 and Table 2 show the effect of Hartmann number on the skin friction $F''(0)$ at fixed $\alpha = -5^{\circ}, \mathrm{Re} = 10$, $\alpha = 5^{\circ}, \mathrm{Re} = 10$ respectively for both cases. It was observed that increase in Hartmann number increases the skin friction.Table 3 and Table 4 present the effect of Reynolds number on the skin friction $F''(0)$ at fixed $\alpha = -5^{\circ}, Ha = 0$, and $\alpha = 5^{\circ}, Ha = 0$ respectively for both cases. It was discovered that increase in Reynolds number increases the skin friction for convergent channel and decreases for divergent channel respectively. There is increase in velocity of the problem as Hartmann number increases at $\mathrm{Re} = 50$ for both convergent and divergent channels respectively as shown in figure 1 and figure 2. If $\mathrm{Re} = 0$ then same result will be gotten for both channels.

VI. CONCLUSIONS

In this article, the nonlinear boundary value problem MHD Jeffery-Hamel flow have been solved using weighted residual method taken into consideration two cases of trial function. Generally, desirable results was gotten using the method with freedom of chosen any trial function that satisfy the boundary condition or force the boundary condition to satisfy the trial function. It was observed that case 2 gave better results than case 1 when compared with the exact solution which make the method reliable for solving nonlinear problems.

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